

Optimal Time Evolution for Hermitian and Non-Hermitian Hamiltonians

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The shortest path between two truths in the real domain passes through the complex domain. — Jacques Hadamard, *The Mathematical Intelligencer* **13** (1991)

Consider the set of all Hamiltonians whose largest and smallest energy eigenvalues, E_{\max} and E_{\min} , differ by a fixed energy ω . Given two quantum states, an initial state $|\psi_I\rangle$ and a final state $|\psi_F\rangle$, there exist many Hamiltonians H belonging to this set under which $|\psi_I\rangle$ evolves in time into $|\psi_F\rangle$. Which Hamiltonian transforms the initial state to the final state in the least possible time τ ? For Hermitian Hamiltonians, τ has a nonzero lower bound. However, among complex non-Hermitian PT -symmetric Hamiltonians satisfying the same energy constraint, τ can be made arbitrarily small without violating the time-energy uncertainty principle. The minimum value of τ can be made arbitrarily small because for PT -symmetric Hamiltonians the evolution path from the vector $|\psi_I\rangle$ to the vector $|\psi_F\rangle$, as measured using the Hilbert-space metric appropriate for this theory, can be made arbitrarily short. The mechanism described here resembles the effect in general relativity in which two space-time points can be made arbitrarily close if they are connected by a wormhole. This result may have applications in quantum computing.

I. INTRODUCTION

Interest in optimal time evolution dates back to the end of the seventeenth century, when the famous brachistochrone problem was solved almost simultaneously by Newton, Leibniz, l'Hôpital, and Jacob and Johann Bernoulli. The word *brachistochrone* is derived from Greek and means shortest time (of flight). The classical brachistochrone problem is stated as follows: A bead slides down a frictionless wire from point A to point B in a homogeneous gravitational field. What is the shape of the wire that minimises the time of flight of the bead? The solution to this problem is that the optimal (fastest) time evolution is achieved when the wire takes the shape of a cycloid, which is the curve that is traced out by a point on a wheel that is rolling on flat ground.

In the past few years there has been much interest in the *quantum brachistochrone* problem, which is formulated in a somewhat similar fashion: Consider two fixed quantum states, an initial state $|\psi_I\rangle$ and a final state $|\psi_F\rangle$ in a Hilbert space. We then consider the set of all Hamiltonians satisfying the energy constraint that the difference between the largest and smallest eigenvalues is a fixed energy ω : $E_{\max} - E_{\min} = \omega$. Some of the Hamiltonians in this set allow the initial state $|\psi_I\rangle$ to evolve into the final state $|\psi_F\rangle$ in time t :

$$|\psi_F\rangle = e^{-iHt/\hbar}|\psi_I\rangle. \quad (1)$$

The quantum brachistochrone problem is to find the *optimal* Hamiltonian; that is, the Hamiltonian that accomplishes this evolution in the shortest possible time, which we denote by τ .

In this chapter we show that for Hermitian Hamiltonians the shortest evolution time τ is a nonzero quantity whose size depends on the Hilbert-space distance between the fixed initial and final state vectors. However, for complex non-Hermitian Hamiltonians, the value of τ can be made arbitrarily small. Thus, non-Hermitian Hamiltonians permit arbitrarily fast time evolution.

Of course, a non-Hermitian Hamiltonian may be physically unrealistic because it may possess complex eigenvalues and/or it may generate nonunitary time evolution; that is, time evolution in which probability is not conserved. However, there is a special class of non-Hermitian Hamiltonians that are *PT* symmetric; that is, Hamiltonians that are invariant under combined space and time reflection. Although such Hamiltonians are not Hermitian in the Dirac sense, they *do* have entirely real spectra and give rise to unitary time evolution. Thus, such Hamiltonians define consistent and acceptable theories of quantum mechanics. We show in this chapter that if we use Hamiltonians of this type to solve the quantum brachistochrone problem, we can achieve arbitrarily fast time evolution without violating any principles of quantum mechanics. Thus, if it were possible to implement faster-than-Hermitian time evolution, then non-Hermitian Hamiltonians might have important applications in quantum computing.

This chapter is organised as follows: In Sec. II we introduce and describe *PT* quantum mechanics and explain how a Hamiltonian that is not Dirac Hermitian can still define a consistent theory of quantum mechanics. Then in Sec. III we explain why complex classical mechanics allows for faster-than-conventional time evolution. In Sec. IV we discuss the quantum brachistochrone for Hermitian Hamiltonians. Then, in Sec. V we extend the discussion in Sec. IV to Hamiltonians that are not Dirac Hermitian. In Sec. VI we explain how it might be possible for a complex Hamiltonian to achieve faster-than-Hermitian time evolution.

II. *PT* QUANTUM MECHANICS

Based on the training that one receives in a traditional quantum mechanics course, one would expect a theory defined by a non-Hermitian Hamiltonian to be physically unacceptable because the energy levels would most likely be complex and the time evolution would most likely be nonunitary (not probability-conserving). However, theories defined by a special class of non-Hermitian Hamiltonians called *PT*-symmetric Hamiltonians can have positive real energy levels and can exhibit unitary time evolution. Such theories are consistent quantum theories. It may be possible to distinguish these theories experimentally from theories defined by Hermitian Hamiltonians because, in principle, non-Hermitian Hamiltonians can be used to generate arbitrarily fast time evolution.

We use the following terminology in this chapter: By *Hermitian*, we mean *Dirac* Hermitian, where the Dirac Hermitian adjoint symbol † represents combined matrix transposition and complex conjugation. The *parity operator* P performs spatial reflection and thus in quantum mechanics it changes the sign of the position operator x and the momentum operator p : $PxP = -x$ and $PpP = -p$. Because the parity operator P is a reflection operator, its square is the unit operator: $P^2 = 1$. The *time-reversal operator* T performs the time reflection $t \rightarrow -t$, and thus it changes the sign of the momentum operator p , $TpT = -p$,

but it leaves the position operator invariant: $TxT = x$. The square of T is the unit operator $T^2 = \mathbf{1}$. We require that the operators P and T individually leave invariant the fundamental Heisenberg algebra of quantum mechanics $[x, p] = i$. Thus, while P is a linear operator, we see that T must perform *complex conjugation* $TzT = z^*$, and hence T is an *antilinear* operator.[62]

The first class of PT -symmetric quantum-mechanical Hamiltonians was introduced in 1998 [6]. Since then there have been many papers on this subject by a wide range of authors. There have also been three recent review articles [4, 5, 29]. In Ref. [6] it was discovered that even if a Hamiltonian is not Hermitian, its energy levels can be all real and positive so long as the eigenfunctions are symmetric under PT reflection.

These new kinds of Hamiltonians are obtained by deforming ordinary Hermitian Hamiltonians into the complex domain. The original class of PT -symmetric Hamiltonians that was proposed in Ref. [6] has the form

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon > 0), \quad (2)$$

where ϵ is a real deformation parameter. Two particularly interesting special cases are obtained by setting $\epsilon = 1$ to get $H = p^2 + ix^3$ and by setting $\epsilon = 2$ to get $H = p^2 - x^4$. Surprisingly, these Hamiltonians have real, positive, discrete energy levels even though the potential for $\epsilon = 1$ is imaginary and the potential for $\epsilon = 2$ is upside-down. The first complete proof of spectral reality and positivity for H in (2) was given by Dorey *et al.* in Ref. [27, 28].

The philosophical background of PT quantum mechanics is simply this: One of the axioms of quantum mechanics requires that the Hamiltonian H be Dirac Hermitian. This axiom is distinct from all other quantum-mechanical axioms because it is mathematical rather than physical in character. The other axioms of quantum mechanics are stated in physical terms; these other axioms require locality, causality, stability and uniqueness of the vacuum state, conservation of probability, Lorentz invariance, and so on. The condition of Dirac Hermiticity $H = H^\dagger$ is mathematical, but the condition of PT symmetry $H = H^{PT} = (PT)H(PT)$ (space-time reflection symmetry) is physical because P and T are elements of the Lorentz group.

The spectrum of H in (2) is real, which poses the question of whether this Hamiltonian specifies a *quantum-mechanical* theory. That is, is the theory specified by H associated with a Hilbert space endowed with a positive inner product and does H specify unitary (norm-preserving) time evolution? The answer to these questions is *yes*. Positivity of the inner product and unitary time evolution was established in Ref. [11, 12] for quantum-mechanical systems having an unbroken PT symmetry (an analogous result was obtained by Mostafazadeh in Ref. [50]) and in Ref. [13] for quantum field theory.

To demonstrate that the theory specified by the H in (2) is a quantum-mechanical theory, we construct a linear operator C that satisfies the three simultaneous algebraic equations [11]: $C^2 = \mathbf{1}$, $[C, PT] = 0$, and $[C, H] = 0$. Using C , which in quantum field theory is a Lorentz scalar [8], we can then construct the appropriate inner product for a PT -symmetric Hamiltonian: $\langle a|b \rangle \equiv a^{CPT} \cdot b$. This inner product, which uses the CPT adjoint, has a strictly positive norm: $\langle a|a \rangle > 0$. Because H commutes with both PT and C , H is *self-adjoint* with respect to CPT conjugation. Also, the time-evolution operator $e^{-iHt/\hbar}$ is unitary with respect to CPT conjugation. Note that the Hilbert space and the CPT inner product are *dynamically determined* by the Hamiltonian itself.

We have explained why a PT -symmetric Hamiltonian gives rise to a unitary theory, but in doing so we raise the question of whether PT -symmetric Hamiltonians are useful. The

answer to this question is simply that PT -symmetric Hamiltonians have *already* been useful in many areas of physics. For example, in 1959 Wu showed that the ground state of a Bose system of hard spheres is described by a non-Hermitian Hamiltonian [60]. Wu found that the ground-state energy of this system is real and he conjectured that all of the energy levels were real. Hollowood showed that the non-Hermitian Hamiltonian for a complex Toda lattice has real energy levels [41]. Cubic non-Hermitian Hamiltonians of the form $H = p^2 + ix^3$ (and also cubic quantum field theories having an imaginary self-coupling term) arise in studies of the Lee-Yang edge singularity [24, 25, 31, 61] and in various Reggeon field-theory models [22, 38, 39]. In all of these cases a non-Hermitian Hamiltonian having a real spectrum appeared mysterious at the time, but now the explanation is clear: In every case the non-Hermitian Hamiltonian is PT symmetric. Hamiltonians having PT symmetry have also been used to describe magnetohydrodynamic systems [34, 37] and to study nondissipative time-dependent systems interacting with electromagnetic fields [30].

An important application of PT quantum mechanics is in the revitalisation of theories that have been thought to be dead because they appear to have ghosts. *Ghosts* are states having negative norm. We have explained above that in order to construct the quantum-mechanical theory defined by a PT -symmetric Hamiltonian, we must construct the appropriate adjoint from the C operator. Having constructed the CPT adjoint, one may find that the so-called ghost state is actually not a ghost at all because when its norm is calculated using the appropriate definition of the adjoint, the norm turns out to be positive. This is what happens in the case of the Lee model.

The Lee model was proposed in 1954 as a quantum field theory in which mass, wavefunction, and charge reorganisation could be performed exactly and in closed form [47]. However, in 1955 Källén and Pauli showed that when the renormalised coupling constant is larger than a critical value, the Hamiltonian becomes non-Hermitian (in the Dirac sense) and a ghost state appears [44]. The importance of the work of Källén and Pauli was emphasised by Salam in his review of their paper [57] and the appearance of the ghost was assumed to be a fundamental defect of the Lee model. However, in 2005 it was shown that the non-Hermitian Lee-model Hamiltonian is PT symmetric and when the norms of the states of this model are determined using the C operator, which can be calculated exactly and in closed form, the ghost state is seen to be an ordinary physical state having positive norm [9]. Thus, the following assertion by Barton [3] is *not correct*: “A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation.”

Another example of a quantum model that was thought to have ghost states, but in fact does not, is the Pais-Uhlenbeck oscillator model [16, 17, 18]. This model has a fourth-order field equation, and for the past several decades it was thought (incorrectly) that all such higher-order field equations lead inevitably to ghosts. It is shown in Ref. [16] that when the Pais-Uhlenbeck model is quantised using the methods of PT quantum mechanics, it does not have any ghost states at all.

There are many potential applications for PT quantum mechanics in areas such as particle physics, cosmology, gravitation, quantum field theory, and solid-state physics. These applications are discussed in detail in the recent review article in Ref. [5]. Furthermore, there are now indications that theories described by PT -symmetric Hamiltonians can be observed in table-top experiments [48, 54, 55].

Having shown the validity and potential usefulness of PT quantum mechanics, one may

ask why PT quantum mechanics works. The reason is that CP is a positive operator, and thus it can be written as the exponential of another operator Q : $CP = e^Q$. The square root of e^Q can then be used to construct a new Hamiltonian \tilde{H} via a similarity transformation on the PT -symmetric Hamiltonian H : $\tilde{H} \equiv e^{-Q/2} H e^{Q/2}$. The new Hamiltonian \tilde{H} has the same energy eigenvalues as the original Hamiltonian H because a similarity transformation is isospectral. Moreover, \tilde{H} is *Dirac Hermitian* [51]; PT quantum mechanics works because there is an isospectral equivalence between a non-Hermitian PT -symmetric Hamiltonian and a conventional Dirac Hermitian Hamiltonian.

There are a number of elementary examples of this equivalence, but a nontrivial illustration is provided by the Hamiltonian H in (2) at $\epsilon = 2$. This Hamiltonian is not Hermitian because boundary conditions that violate the L^2 norm must be imposed in Stokes wedges in the complex plane in order to obtain a real, positive, discrete spectrum. The exact equivalent Hermitian Hamiltonian is $\tilde{H} = p^2 + 4x^4 - 2\hbar x$, where \hbar is Planck's constant [10, 23, 43]. The term proportional to \hbar vanishes in the classical limit and is thus an example of a quantum anomaly.

Since PT symmetry is equivalent by means of a similarity transformation to conventional Dirac Hermiticity, one may wonder whether PT quantum mechanics is actually fundamentally different from ordinary quantum mechanics. The answer is *yes* and, at least in principle, there is an experimentally observable difference between PT -symmetric and ordinary Dirac Hermitian Hamiltonians. The quantum brachistochrone provides a setting for examining this difference and provides a way to discriminate between the class of PT -symmetric Hamiltonians and the class of Dirac Hermitian Hamiltonians.

III. COMPLEX CLASSICAL MOTION

It is implicitly assumed in the derivation of the classical brachistochrone that the path of shortest time of descent is *real*. However, it is interesting that if one allows for the possibility of complex paths of motion, one can achieve an even shorter time of flight. In this section we consider a simple classical-mechanical system. Our purpose is to explain heuristically how extending a dynamical system into the complex domain can result in faster-than-real time evolution.

To demonstrate that a shorter time of flight can be achieved by means of complex paths, let us consider the classical harmonic oscillator, whose Hamiltonian is

$$H = p^2 + x^2. \quad (3)$$

If a particle has energy $E = 1$, then the classical turning points of the motion of the particle are located at $x = \pm 1$. The particle undergoes simple harmonic motion in which it oscillates sinusoidally between these two turning points. This periodic motion is indicated in Fig. 1 by a solid line connecting the turning points. However, in addition to this oscillatory motion on the real- x axis, there are an infinite number of other trajectories that a particle of energy E can have [7]. These classical trajectories, which are also shown in Fig. 1, are all ellipses whose foci are located at precisely the positions of the turning points. All of the classical orbits are periodic, and all orbits have the same period $T = 2\pi$. Thus, a classical particle travels faster along more distant ellipses.

Now suppose that a classical particle of energy $E = 1$ travels along the real- x axis from some point $x = -a$ to $x = a$, where $a > 1$. If the potential $V(x)$ is everywhere zero along its

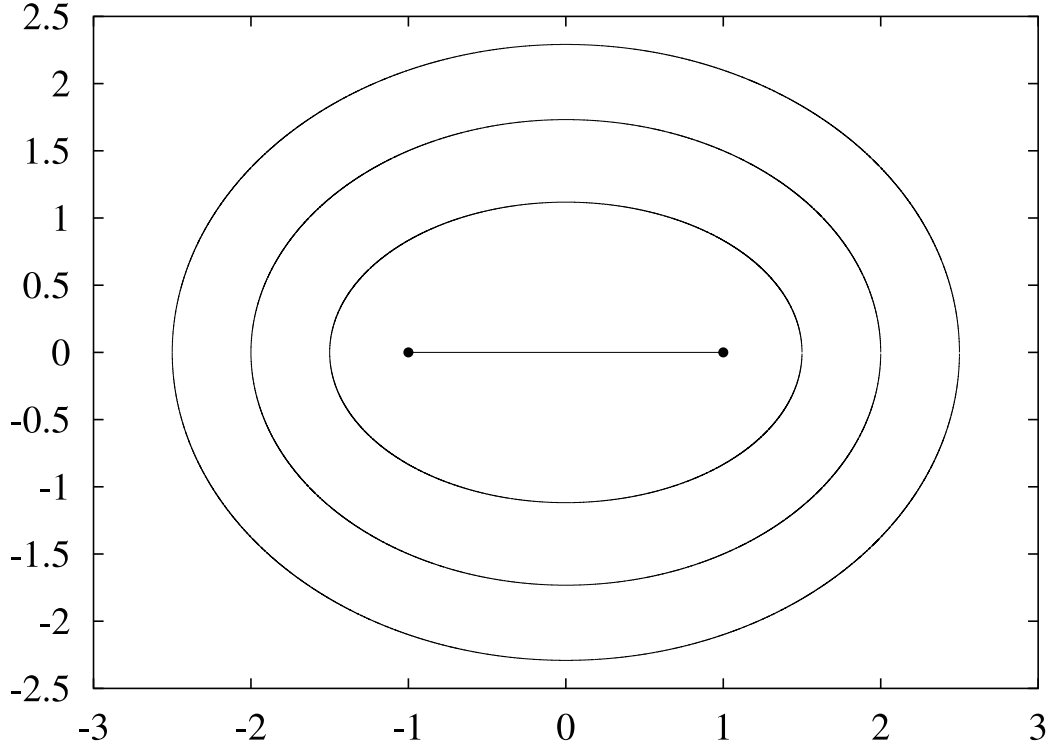


FIG. 1: Classical trajectories in the complex- x plane for the harmonic oscillator whose Hamiltonian is $H = p^2 + x^2$. These trajectories represent the possible paths of a particle whose energy is $E = 1$. The trajectories are nested ellipses with foci located at the turning points at $x = \pm 1$. The real line segment (degenerate ellipse) connecting the turning points is the usual periodic classical solution to the harmonic oscillator. All closed paths have the same period 2π .

path, then it will travel at a constant velocity. However, suppose that the particle suddenly finds itself in the parabolic potential $V(x) = x^2$ when it reaches the turning point at $x = -1$ and that it suddenly escapes the influence of this potential at $x = 1$. Then, the time of flight from $x = -a$ to $x = a$ will be changed because the particle does not travel at a constant velocity between the turning points. Next, let us imagine that the potential $V(x) = x^2$ is suddenly turned on *before* the particle reaches the turning point at $x = -1$. In this case, the particle will follow one of the elliptical paths in the complex plane around to the positive real axis. Just as the particle reaches the positive real axis the potential is turned off, so the particle proceeds onward along the real axis until it reaches $x = a$. This trip will take less time because the particle travels faster along the ellipse in the complex plane.

We have arrived at the surprising conclusion that if the classical particle enters the parabolic potential $V(x) = x^2$ immediately after it begins its voyage up the real axis, its time of flight will be exactly half a period, or π . Indeed, by travelling in the complex plane, a particle of energy $E = 1$ can go from the point $x = -a$ to the point $x = a$ in time π , no matter how large a is. Evidently, if a particle is allowed to follow complex classical trajectories, then it is possible to make a drastic reduction in its time of flight between two given real points.

IV. HERMITIAN QUANTUM BRACHISTOCHRONE

The quantum brachistochrone problem, as described briefly in Sec. I, is similar to the classical counterpart except that the optimisation takes place in a Hilbert space. Specifically, we are given a pair of quantum states, an initial state $|\psi_I\rangle$ and the final state $|\psi_F\rangle$, and we would like to find the one-parameter family of unitary operators $\{U_t\}$ that achieves the transformation $|\psi_I\rangle \rightarrow |\psi_F\rangle = U_t|\psi_I\rangle$ in the smallest possible time t . Since a one-parameter family of unitary operators can be formed in terms of a Hermitian operator H as $U_t = \exp(-iHt/\hbar)$, the problem is equivalent to finding the Hermitian operator H that realises the transformation $|\psi_I\rangle \rightarrow |\psi_F\rangle$ in the shortest possible time.

The Hermitian operator H can be thought of as representing the Hamiltonian, so the quantum brachistochrone problem is equivalent to finding the optimal Hamiltonian H satisfying $\exp(-iHt/\hbar)|\psi_I\rangle = |\psi_F\rangle$. However, it is intuitively clear that if we are allowed to have access to an unbounded energy resource, then the time required for the relevant transformation, irrespective of whether the Hamiltonian is optimal or not, can be made arbitrary small. Hence, for a quantum brachistochrone problem to possess a nontrivial solution, some form of constraint is needed. The simplest constraint is to assume that the energy is bounded so that the difference between the largest and the smallest energy eigenvalues has a fixed value ω :

$$E_{\max} - E_{\min} = \omega. \quad (4)$$

A short calculation shows that if the Hamiltonian H is bounded, then (i) the standard deviation of the energy is bounded according to

$$\Delta H \leq \frac{1}{2}(E_{\max} - E_{\min}), \quad (5)$$

and (ii) the state with maximum energy uncertainty is $(|E_{\max}\rangle + |E_{\min}\rangle)/\sqrt{2}$. It follows that the energy constraint (4) is equivalent to a constraint on energy uncertainty.

The brachistochrone problem of this type has been analysed recently and a solution was obtained by means of a variational method [26]. It has also been solved in terms of a more elementary approach making use of the geometry of quantum state space [20]. We shall follow closely the latter approach here.

Let us now state the simplest form of the quantum brachistochrone problem: We have a quantum system represented by an N -dimensional Hilbert space \mathcal{H} and a prescribed pair of states $|\psi_I\rangle$ and $|\psi_F\rangle$ on \mathcal{H} . The problem is (a) to find the Hamiltonian H satisfying the constraint (4) such that the unitary transformation $\exp(-iHt/\hbar)|\psi_I\rangle = |\psi_F\rangle$ is achieved in shortest possible time and (b) to find the time required to realise such an operation.

A little geometric intuition allows us to find the solution to this problem with minimum effort. Recall that the time required for transporting a state along a path in \mathcal{H} is given by the *distance* divided by the *speed*. Hence, all we have to do is first to identify the shortest path and measure its length and then to allow the state to evolve along the path with the greatest possible speed without violating the constraint (4).

In quantum mechanics the notion of distance is closely linked to the notion of transition probability [21, 42]. In particular, by looking at the transition probability between neighbouring states we can derive the expression for the metric on the space of quantum states. This allows us to measure distances between states. The idea can be sketched as follows: Consider a state $|\psi\rangle$ in \mathcal{H} and a neighbouring state $|\psi\rangle + |\mathrm{d}\psi\rangle$. The transition probability

between these states is

$$\cos^2 \left(\frac{1}{2} ds \right) = \frac{(\langle \psi | + \langle d\psi |) | \psi \rangle \langle \psi | (| \psi \rangle + | d\psi \rangle)}{\langle \psi | \psi \rangle (\langle \psi | + \langle d\psi |) (| \psi \rangle + | d\psi \rangle)}, \quad (6)$$

where ds defines the line element on the space of pure states. By using

$$\cos^2 \left(\frac{1}{2} ds \right) \approx 1 - \frac{1}{4} ds^2, \quad (7)$$

expanding the right side of (6), and retaining terms of quadratic order, we find that the line element is

$$ds^2 = 4 \frac{\langle \psi | \psi \rangle \langle d\psi | d\psi \rangle - \langle \psi | d\psi \rangle \langle d\psi | \psi \rangle}{\langle \psi | \psi \rangle^2}. \quad (8)$$

This line element is known in geometry to arise from the *Fubini-Study metric* [46] and it can be used to measure the distance of the shortest path joining a pair of points on the space of pure quantum states.

If the Hilbert space is two dimensional, then a generic normalised state vector $|\psi\rangle$ can be expressed in the form

$$|\psi\rangle = \begin{pmatrix} \cos \frac{1}{2}\theta \\ \sin \frac{1}{2}\theta e^{i\phi} \end{pmatrix}. \quad (9)$$

A short calculation then shows that the Fubini-Study line element (8) reduces in this case to the expression

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad (10)$$

which we recognise as the line element on the Bloch sphere \mathcal{S} . (The Bloch sphere is the state space of two-level systems.)

In the case of an n -dimensional Hilbert space \mathcal{H} , if we are given a pair of distinct states $|\psi_I\rangle$ and $|\psi_F\rangle$, then the linear span of these two states forms a two-dimensional subspace of \mathcal{H} . It should be intuitively clear that the shortest path joining $|\psi_I\rangle$ and $|\psi_F\rangle$ should lie on this two-dimensional subspace. Thus, irrespective of the dimensionality of \mathcal{H} *the solution to our quantum brachistochrone problem can be obtained by analysing the two-dimensional subspace spanned by $|\psi_I\rangle$ and $|\psi_F\rangle$* . Even when we restrict our attention to this subspace, there still remain infinitely many unitary orbits that realise the transformation $|\psi_I\rangle \rightarrow |\psi_F\rangle = U_t |\psi_I\rangle$. However, since the two-dimensional state space is just the Bloch sphere \mathcal{S} endowed with the spherical metric (10), we see that there is a unique great circle arc that joins $|\psi_I\rangle$ and $|\psi_F\rangle$ on \mathcal{S} . (This assumes, of course, that $|\psi_I\rangle$ and $|\psi_F\rangle$ are not antipodal points of the sphere. Otherwise, there are infinitely many such paths.) In this way we have identified the shortest path joining $|\psi_I\rangle$ and $|\psi_F\rangle$. The shortest distance s_{\min} between these two points of \mathcal{S} is thus given by

$$s_{\min} = 2 \arccos \left(\frac{|\langle \psi_I | \psi_F \rangle|}{\sqrt{\langle \psi_I | \psi_I \rangle \langle \psi_F | \psi_F \rangle}} \right). \quad (11)$$

This result can also be obtained by integrating the line element (10) along the great-circle arc on \mathcal{S} .

Having obtained the distance of the shortest path we proceed to find the maximum speed at which the state can evolve unitarily. For the evolution of the state we must consider the general Schrödinger equation, but we also need to express the equation in the correct form. This is the so-called modified Schrödinger equation

$$\frac{d|\psi_t\rangle}{dt} = -\frac{i}{\hbar} \tilde{H}|\psi_t\rangle. \quad (12)$$

In this equation the mean-adjusted Hamiltonian \tilde{H} is given by

$$\tilde{H} = H - \langle H \rangle, \quad (13)$$

where

$$\langle H \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}. \quad (14)$$

Note that $\langle \tilde{H} \rangle = 0$ and that according to (12) the tangent vector $\frac{d}{dt}|\psi_t\rangle$ is everywhere orthogonal to the direction of the state [45]. Since the energy expectation $\langle H \rangle$ depends on the state $|\psi\rangle$, the modified Schrödinger equation appears to be nonlinear. However, this is not the case. The point is that the expectation value of the Hamiltonian is a constant of the motion under the Schrödinger dynamics. Thus, given the initial state $|\psi_I\rangle$, we calculate $\langle H \rangle$ in this state and subtract this number from the Hamiltonian. Since the Hamiltonian in quantum mechanics is defined only up to an additive constant, this modification does not alter the physics in any way. It is worthwhile noting that the modified Schrödinger equation (12) is canonical and reduces to the standard eigenvalue problem when the state $|\psi_t\rangle$ is time independent without one having to evoke the correspondence principle [21].

If the initial state vector $|\psi_I\rangle$ is normalised, then the evolution (12) preserves the norm. It follows that $\langle \psi | d\psi \rangle = 0$. Since the speed v of quantum evolution is given by $v = ds/dt$, we find from (8) and (12) that

$$v^2 = \frac{4}{\hbar^2} \langle \psi_t | (H - \langle H \rangle)^2 | \psi_t \rangle = \frac{4}{\hbar^2} \langle \psi_I | (H - \langle H \rangle)^2 | \psi_I \rangle. \quad (15)$$

This shows that the speed of quantum evolution is given by the energy uncertainty. The expression (15) for the speed of quantum evolution is known as the *Anandan-Aharonov relation* [1]. Since we know from (5) that under the constraint (4) the energy uncertainty ΔH is bounded by $\frac{1}{2}\omega$, we find that the maximum speed of quantum evolution is given by

$$v_{\max} = \frac{\omega}{\hbar}. \quad (16)$$

By using the results in (11) and (16) we deduce that the minimum time required for realising the unitary transportation $|\psi_I\rangle \rightarrow |\psi_F\rangle = U_t|\psi_I\rangle$ is given by the ratio s_{\min}/v_{\max} . In particular, if $|\psi_I\rangle$ and $|\psi_F\rangle$ are orthogonal, then they correspond to antipodal points on the Bloch sphere \mathcal{S} , and we have $s_{\min} = \pi$. In this case, the minimum time required to orthogonalize the state (that is, for the state to evolve into a new state that is orthogonal to the original state) is known as the *passage time* τ_P [19, 58]. The passage time is explicitly

$$\tau_P = \frac{\pi\hbar}{2\Delta H} = \frac{\pi\hbar}{\omega}. \quad (17)$$

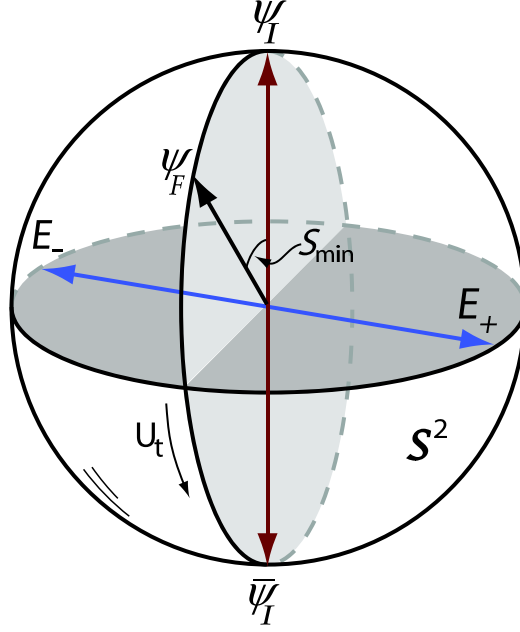


FIG. 2: Optimal Hamiltonian for quantum state transportation. The two-dimensional complex Hilbert space spanned by the initial state $|\psi_I\rangle$ and the final state $|\psi_F\rangle$ can be visualised in real terms as a Bloch sphere \mathcal{S} . The two states $|\psi_I\rangle$ and $|\psi_F\rangle$ can then be identified as a pair of points on \mathcal{S} . Assuming that these two points are not antipodal, there exists a unique great circle arc joining these two points, which determines the shortest path joining the two states. The optimal way of unitarily transporting $|\psi_I\rangle$ into $|\psi_F\rangle$ is therefore to rotate the sphere along the axis orthogonal to the great circle. The axis of rotation then specifies two quantum states: $|E_-\rangle$ and $|E_+\rangle$. These states are the eigenstates of the optimal Hamiltonian H .

The passage time (17) provides the bound in Hermitian quantum mechanics for transporting a state into an orthogonal state, and is sometimes referred to as the *Fleming bound* [32].

The ratio s_{\min}/v_{\max} gives the solution to part (b) of our quantum brachistochrone problem. To solve part (a), that is, to find the optimal Hamiltonian, we argue as follows: Since the problem is confined to a two-dimensional subspace of \mathcal{H} , the solution can be obtained by elementary trigonometry on the Bloch sphere \mathcal{S} . The key idea is to recall that the shortest path joining $|\psi_I\rangle$ and $|\psi_F\rangle$ is a great circle arc on \mathcal{S} . Without loss of generality, we can assume that $|\psi_I\rangle$ and $|\psi_F\rangle$ lie on the equator of \mathcal{S} with respect to a suitable choice of axis. Then, the unitary motion along the shortest path can be generated by the rotation of \mathcal{S} along this axis. Since the eigenstates of the Hamiltonian H that generates such a rotation correspond to the antipodal points along this axis, the pair of states $|\psi_I\rangle$ and $|\psi_F\rangle$ can both be expressed as equal superpositions of the eigenstates of H with the relative phase shifted by s_{\min} . Writing $|E_+\rangle$ and $|E_-\rangle$ for the normalised eigenstates of H and using $\alpha = s_{\min}/2$, we can express the initial and the final state in the form

$$\frac{1}{\sqrt{2}}(|E_-\rangle + e^{-i\alpha}|E_+\rangle) = |\psi_I\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}}(|E_-\rangle + e^{i\alpha}|E_+\rangle) = |\psi_F\rangle. \quad (18)$$

Solving these equations for $|E_+\rangle$ and $|E_-\rangle$, we obtain

$$|E_-\rangle = \frac{i}{\sqrt{2} \sin \alpha} (e^{-i\alpha}|\psi_F\rangle - e^{i\alpha}|\psi_I\rangle) \quad (19)$$

and

$$|E_+\rangle = -\frac{i}{\sqrt{2}\sin\alpha}(|\psi_F\rangle - |\psi_I\rangle). \quad (20)$$

These are the eigenstates of the optimal Hamiltonian H that generates the unitary motion $|\psi_I\rangle \rightarrow |\psi_F\rangle = U_t|\psi_I\rangle$ along the shortest path. The eigenvalues of the optimal H can be arbitrary as long as the condition (4) is satisfied. Without loss of generality, we may assume H be trace free, and we obtain the solution to part (a) of the quantum brachistochrone problem:

$$H = \frac{1}{2}\omega|E_+\rangle\langle E_+| - \frac{1}{2}\omega|E_-\rangle\langle E_-|. \quad (21)$$

This is the ‘minimal’ solution to the problem in the sense that H acts only on the two-dimensional subspace of \mathcal{H} while leaving the rest of \mathcal{H} unperturbed.

As a special example, consider the problem of a spin-flip, that is, turning a spin-up state into a spin-down state unitarily. In this case the initial and the final states can be written as

$$|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_F\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (22)$$

in the spin- z basis. Substituting these into (19) and (20), we find that the eigenstates of the optimal Hamiltonian are

$$|E_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |E_+\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (23)$$

because in this case we have $\alpha = \pi/2$. Substituting this result into (21) yields

$$H = \frac{1}{2}\begin{pmatrix} 0 & -\omega \\ -\omega & 0 \end{pmatrix} \quad (24)$$

for the optimal Hamiltonian. By using the relation

$$e^{i\phi\sigma\cdot\mathbf{n}} = \cos\phi\mathbb{1} + i\sin\phi\sigma\cdot\mathbf{n}, \quad (25)$$

where \mathbf{n} is a unit vector and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (26)$$

are Pauli matrices, we obtain the expression for the optimal unitary operator:

$$U_t = \begin{pmatrix} \cos\left(\frac{\omega t}{2\hbar}\right) & -i\sin\left(\frac{\omega t}{2\hbar}\right) \\ -i\sin\left(\frac{\omega t}{2\hbar}\right) & \cos\left(\frac{\omega t}{2\hbar}\right) \end{pmatrix}. \quad (27)$$

It follows at once that the optimal unitary orbit $|\psi_t\rangle = U_t|\psi_I\rangle$ is given by

$$|\psi_t\rangle = \begin{pmatrix} \cos\left(\frac{\omega t}{2\hbar}\right) \\ -i\sin\left(\frac{\omega t}{2\hbar}\right) \end{pmatrix}. \quad (28)$$

We find that the first time at which $|\psi_t\rangle$ reaches $|\psi_F\rangle$ is given by the condition $\omega t/2\hbar = \pi/2$, that is, when $t = \tau_P$, where τ_P is the passage time given in (17).

We have seen how the simplest form of a quantum brachistochrone problem can be solved in Hermitian quantum mechanics by considering a two-dimensional Hilbert subspace combined with elementary geometric constructions on it. In a more general situation the unitary motion may be constrained further so that the optimal Hamiltonian (21) may not be implementable. For example, the constraint may enforce the path of the unitary evolution to lie in a three- rather than in a two-dimensional subspace. To determine what happens let us work out the passage time for this example.

Since in this case the minimal solution H to the brachistochrone problem is a three-dimensional matrix, we can express the initial state $|\psi_I\rangle$ in terms of the three eigenstates of H according to

$$|\psi_I\rangle = \cos \alpha |E_i\rangle + \sin \alpha \cos \beta e^{i\phi} |E_j\rangle + \sin \alpha \sin \beta e^{i\varphi} |E_k\rangle, \quad (29)$$

where α, β are angular coordinates, ϕ, φ are phase variables, and we assume that $E_i < E_j < E_k$. If a unitary operator U_T transforms this state into an orthogonal state, then the condition

$$\cos^2 \alpha + \sin^2 \alpha \cos^2 \beta e^{-i\omega_{ji}T/\hbar} + \sin^2 \alpha \sin^2 \beta e^{-i\omega_{ki}T/\hbar} = 0 \quad (30)$$

must be satisfied, where $\omega_{ji} = E_j - E_i$ and $\omega_{ki} = E_k - E_i$. To render the analysis more tractable, we simplify this constraint by assuming that $\alpha = \beta = \pi/4$. Then (30) implies that a necessary condition for the state $|\psi_I\rangle$ to evolve into an orthogonal state is given by the relation

$$\frac{\omega_{ki}}{\omega_{ji}} = \frac{2m-1}{2n-1}, \quad (31)$$

where m, n are natural numbers such that $m \neq n$.

Thus, the solution to the brachistochrone problem must be such that the eigenvalues of H satisfy condition (31) as well as the constraint $E_{\max} - E_{\min} \leq \omega$. Assuming that these constraints are indeed satisfied, the initial state evolves into an orthogonal state $|\psi_F\rangle$. The first time that $|\psi_I\rangle$ evolves into a state orthogonal to $|\psi_I\rangle$, in particular, is given by

$$T = \frac{\pi\hbar}{\omega_{ji}} = \frac{3\pi\hbar}{\omega_{ki}}. \quad (32)$$

However, since in this case $U_t|\psi_I\rangle$ does not describe a geodesic path, T will be larger than Fleming's passage time τ_P given in (17). Indeed, without loss of generality we may set $E_i = 0$. Then, it is straightforward to verify that $T = \sqrt{6}\tau_P$. This follows from the fact that under the constraint $\omega_{ki} = 3\omega_{ji}$ that comes from (32), the squared energy dispersion in the state (29) with $\alpha = \beta = \pi/4$ is given by $\Delta H^2 = \frac{3}{2}\omega_{ji}^2$.

V. NON-HERMITIAN QUANTUM BRACHISTOCHRONE

We have seen how the solution to the simple brachistochrone problem can be obtained in the Hermitian quantum theory. What happens if we extend the quantum theory into the complex domain by looking at a PT -symmetric theory? We saw earlier that in classical

mechanics if we were to allow for a complex path interpolating a pair of real points of the coordinate space, then it is possible (at least mathematically) to transport a particle across a large distance in virtually no time. It turns out that an analogous situation emerges in the PT -symmetric theory. Here we present a simple algebraic calculation of the optimal evolution time from an initial state to a final state by using a simple 2×2 Hamiltonian. As we have remarked above, the 2×2 model suffices to cover general cases because in the case of our simple brachistochrone problem the solution is found on the two-dimensional subspace of the Hilbert space spanned by the initial state $|\psi_I\rangle$ and the final state $|\psi_F\rangle$. In the case of a PT -symmetric Hamiltonian the variational approach gives a more direct way to handle the brachistochrone problem. Thus, we shall first briefly revisit the Hermitian case but expressed in the variational formalism and then we will compare the result to its PT -symmetric counterpart.

A. Hermitian 2×2 matrices

We choose a basis so that the initial and final state vectors take the form

$$|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_F\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad (33)$$

where the condition that $|\psi_F\rangle$ be normalised is $|a|^2 + |b|^2 = 1$. The most general 2×2 Hermitian Hamiltonian is

$$H = \begin{pmatrix} s & r e^{-i\theta} \\ r e^{i\theta} & u \end{pmatrix} \quad (r, s, u, \theta \text{ real}). \quad (34)$$

For this Hamiltonian the eigenvalue constraint (4) takes the form

$$\omega^2 = (s - u)^2 + 4r^2. \quad (35)$$

To find the optimal Hamiltonian satisfying this constraint, we rewrite H as a linear combination of Pauli matrices:

$$H = \frac{1}{2}(s + u)\mathbb{1} + \frac{1}{2}\omega\sigma \cdot \mathbf{n}, \quad (36)$$

where

$$\mathbf{n} = \frac{1}{\omega}(2r \cos \theta, 2r \sin \theta, s - u) \quad (37)$$

is a unit vector. Then by use of the identity (25) the evolution equation $|\psi_F\rangle = e^{-iH\tau/\hbar}|\psi_I\rangle$ can be expressed in the form

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{-\frac{1}{2}i(s+u)t/\hbar} \begin{pmatrix} \cos \frac{\omega t}{2\hbar} - i \frac{s-u}{\omega} \sin \frac{\omega t}{2\hbar} \\ -i \frac{2r}{\omega} e^{i\theta} \sin \frac{\omega t}{2\hbar} \end{pmatrix}. \quad (38)$$

The second component of this equation gives $|b| = \frac{2r}{\omega} \sin \frac{\omega t}{2\hbar}$, which allows us to find the required time of evolution:

$$t = \frac{2\hbar}{\omega} \arcsin \frac{\omega|b|}{2r}. \quad (39)$$

We must now minimise the time t over all $r > 0$ while maintaining the energy constraint in (35). This constraint tells us that the maximum value of r is $\frac{1}{2}\omega$. At this value we have $s = u$. Because H can be made trace free, we can set $s = u = 0$. The variable θ in (36) does not affect the eigenvalues, so we may set $\theta = \pi$. Then we recover the optimal Hamiltonian obtained in (24). As regards the minimum evolution time τ we have

$$\tau = \frac{2\hbar \arcsin |b|}{\omega}. \quad (40)$$

In the special case for which $a = 0$ and $b = 1$ so that $|\psi_I\rangle$ and $|\psi_F\rangle$ are orthogonal, we recover the passage time $\tau = \tau_P = \pi\hbar/\omega$, the smallest time required for a spin flip.

Although the form of the result in (40) resembles the statement of the uncertainty principle, it is merely the statement indicated above that *rate \times time = distance*; the maximum speed of evolution is given by ΔH , and the distance between $|\psi_I\rangle$ and $|\psi_F\rangle$, assuming they are normalised, is given by $2 \arccos(|\langle\psi_F|\psi_I\rangle|)$. Since $|\langle\psi_F|\psi_I\rangle| = |a|$ and $|a| = \sqrt{1 - |b|^2}$, we obtain (40) from the relation

$$\arccos \sqrt{1 - |b|^2} = \arcsin |b|. \quad (41)$$

B. Non-Hermitian 2×2 matrices

We now show by direct calculation that for a PT -symmetric Hamiltonian, τ can be arbitrarily small. This is because a PT -symmetric Hamiltonian whose eigenvalues are all real is equivalent to a Hermitian Hamiltonian via $\tilde{H} = e^{-Q/2} H e^{Q/2}$, where Q is Dirac Hermitian. The states in a PT -symmetric theory are mapped by $e^{-Q/2}$ to the corresponding states in the Dirac Hermitian theory. But, the overlap distance between two states does not remain constant under a similarity transformation. We can exploit this property of the similarity transformation to overcome the Hermitian lower limit on the time τ . (The detailed calculation is explained in Ref. [14].)

We consider the general class of PT -symmetric 2×2 Hamiltonians having the form

$$H = \begin{pmatrix} r e^{i\theta} & s \\ s & r e^{-i\theta} \end{pmatrix} \quad (r, s, \theta \text{ real}). \quad (42)$$

The time reversal operator T performs complex conjugation and the parity operator in this case is given by

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (43)$$

The two eigenvalues

$$E_{\pm} = r \cos \theta \pm \sqrt{s^2 - r^2 \sin^2 \theta} \quad (44)$$

are real if $s^2 > r^2 \sin^2 \theta$. This inequality defines the region of unbroken PT symmetry. The unnormalised eigenstates of H are

$$|E_+\rangle = \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \quad \text{and} \quad |E_-\rangle = \begin{pmatrix} ie^{-i\alpha/2} \\ -ie^{i\alpha/2} \end{pmatrix}, \quad (45)$$

where α is given by $\sin \alpha = (r/s) \sin \theta$. Note that the condition of unbroken PT symmetry of H in (42) implies that α is real. The C operator required for defining the Hilbert space inner product is

$$C = \frac{1}{\cos \alpha} \begin{pmatrix} i \sin \alpha & 1 \\ 1 & -i \sin \alpha \end{pmatrix}. \quad (46)$$

It is easy to verify that the CPT norms of both eigenstates have the value $\sqrt{2 \cos \alpha}$.

To calculate τ we express the Hamiltonian H in (42) as

$$H = (r \cos \theta) \mathbb{1} + \frac{1}{2} \omega \sigma \cdot \mathbf{n}, \quad (47)$$

where

$$\mathbf{n} = \frac{2}{\omega} (s, 0, i r \sin \theta) \quad (48)$$

is a unit vector. The energy constraint requires that the squared difference between energy eigenvalues is

$$\omega^2 = 4s^2 - 4r^2 \sin^2 \theta. \quad (49)$$

The positivity of ω^2 is ensured by the condition of unbroken PT symmetry. Notice that (49) differs from (35) by a sign. We can think of (49) as being *hyperbolic* in character, while (35) is *elliptic* in character. The technical advantage of the constraint in (49) is that because of the minus sign both terms on the right side can become large without violating the condition that ω be fixed. We will see that it is this fact that allows the non-Hermitian Hamiltonian H in (42) to achieve faster-than-Hermitian time evolution.

To determine τ we write down the PT -symmetric time-evolution equation in vector form:

$$e^{-iHt/\hbar} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{-itr \cos \theta/\hbar}}{\cos \alpha} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar} - \alpha) \\ -i \sin(\frac{\omega t}{2\hbar}) \end{pmatrix}. \quad (50)$$

In particular, consider the pair of vectors used in the Hermitian spin-flip case as in (22). Observe that the evolution time t needed to reach $|\psi_F\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ from $|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is given by

$$t = \frac{(2\alpha - \pi)\hbar}{\omega}. \quad (51)$$

Optimising this result over allowable values for α as α approaches $\frac{1}{2}\pi$, the optimal time τ tends to zero, a dramatic change from the Hermitian result in (17)! Note, however, that the two vectors in (22) are not orthogonal with respect to the CPT inner product. This is the reason that the Fleming bound in (17) is not violated.

VI. EXTENSION OF NON-HERMITIAN HAMILTONIANS TO HIGHER-DIMENSIONAL HERMITIAN HAMILTONIANS

We have seen how a quantum state can be transported unitarily into another state in arbitrary short time by using a bounded Hamiltonian if we allow for a complex path interpolating them in the space of unitary motions. Can such an operation be implemented

in practice? If the answer is affirmative, then the implication is immense in quantum information, computation, cryptography, and other related fields. For example, if a quantum computer were to exist, then solutions to difficult optimisation problems can in principle be found in arbitrary short time, and this in turn would have important implications in society as a whole.

A *gedanken* experiment was proposed in Ref. [14] to realise this effect in a laboratory. The setup is as follows: we use a Stern-Gerlach filter to create a beam of spin-up electrons. The beam then passes through a *black box* containing a device governed by a PT -symmetric Hamiltonian that flips the spins unitarily in a very short time. The outgoing beam then enters a second Stern-Gerlach device that verifies that the electrons are now in spin-down states. In effect, the black box device is *applying a complex magnetic field* \vec{B} :

$$\vec{B} = (s, 0, ir \sin \theta). \quad (52)$$

If the field strength has sufficiently large amplitude, then spins can be flipped in virtually no time because the complex path joining these two states is arbitrarily short without violating the energy constraint in (49). We emphasise that the field strength can be made large without violating the energy constraint (4) is a consequence of the hyperbolic representation in (49).

The arbitrarily short alternative complex pathway from an up state to a down state, as illustrated by this thought experiment, is reminiscent of the short alternative distance between two widely separated space-time points as measured through a wormhole in general relativity. This comparison is of course controversial, and it has subsequently motivated much research and it has generated a lively debate in the literature [2, 15, 33, 35, 36, 49, 52, 53, 56]. We emphasise that the entire package of flipping the spin is not realised by a unitary operation. This follows from the fact that PT -symmetric quantum theory is unitary, and as such it respects the Fleming bound (17) applicable to all unitary theories [14]. The point is that the ‘black box’ scheme described above actually consists of three regimes: (i) the preparation of a spin-up state in the Hermitian setup, (ii) the fast unitary motion to flip the spin using a PT -symmetric Hamiltonian, and (iii) the recovery of a spin-down state in the Hermitian setup. Thus, the operation is locally unitary, but the switching between Hermitian and PT -symmetric description is not unitary. This three-step process of switching Hamiltonians is analogous to the classical procedure described in Sec. III for obtaining faster-than-real time evolution. Recall that in the classical case we were able to transport a finite-energy particle across a large distance in a short time by switching the potential through which it was travelling. Note that in the classical case there is no question of violating unitarity because the particle does not get lost.

The question of unitarity and faster-than-Hermitian time evolution has been reexamined in more detail in Ref. [35, 53] by means of a geometric approach and also in Ref. [2], where a more general class of non-Hermitian Hamiltonians that are not necessarily PT symmetric are considered. In particular, Mostafazadeh has emphasised the role of quantum observables in such an experiment; the spin operator in Hermitian quantum mechanics cannot be interpreted as a spin operator in the PT -symmetric counterpart, thus leading to ambiguities regarding the physical interpretation of the *gedanken* experiment described above.

An intriguing alternative proposal for an implementation of the fast spin flip has been made more recently by Günther and Samsonov [36]. The idea is to embed the problem into a Hermitian setup represented by a higher-dimensional Hilbert space. Specifically, take our

PT -symmetric Hamiltonian H in (42). The eigenstates of H are not orthogonal with respect to the Hermitian inner product. Since H is not Hermitian, its Hermitian conjugate defines a new matrix H^\dagger . The eigenstates of H^\dagger thus also define a pair of nonorthogonal states in the Hermitian theory. When these four states are suitably normalised, they can be used to form an over-complete basis set in the Hermitian two-dimensional Hilbert space. Such an over-complete set of basis is also known as a positive operator-valued measure (POVM), commonly used in the analysis of quantum information theory. A key idea is that such a basis can be embedded in a higher-dimensional Hilbert space to form an orthogonal basis by using the Naimark dilation [40]. A Hermitian Hamiltonian can then be constructed—in this case a 4×4 matrix—such that its eigenstates are precisely the four states thus obtained. Using this Hamiltonian it is possible to construct a standard unitary motion in such a way that the induced motion obtained by the projection onto the two-dimensional subspace is characterised by the PT -symmetric motion (50).

In this way, Günther and Samsonov were able to show that the fast spin flip can in principle be realised in the standard Hermitian quantum mechanics by a combination of a unitary motion and a projection in a larger-dimensional Hilbert space. In practical terms this means that one should couple the spin to an auxiliary particle (this can be done either by a projection or by a unitary operation), apply a unitary evolution in the larger Hilbert space of the combined system, and finally project out the auxiliary particle to recover the spin in the transported state. The net effect of such an operation can then be characterised by (50). The Fleming bound is apparently violated due to the general fact that when a unitary motion is projected to a subspace of a Hilbert space, the resulting dynamics need not respect laws of unitarity. It would be of considerable interest to find out whether the Günther and Samsonov scheme can actually be implemented in a laboratory, and if not, what might be the difficulty preventing the violation of the Fleming bound.

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